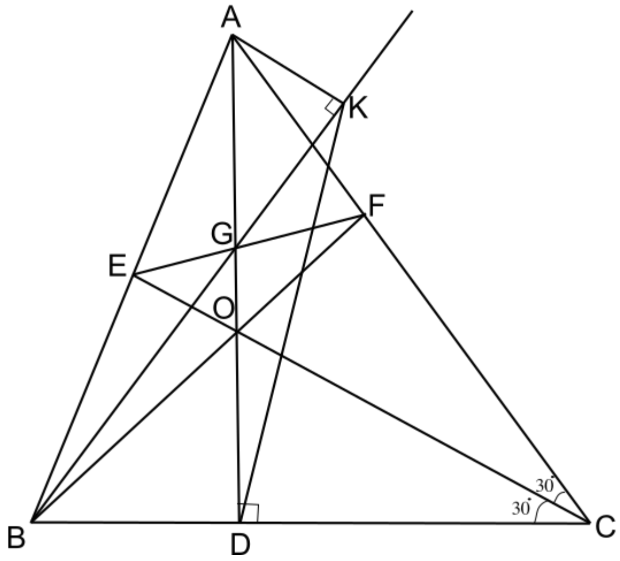
Author's Solution of 01.10.2023 Cash Award Question



In ΔABC, $\angle C = 60°$

And $\angle DAC = 30°$ & $\angle ADC = 90°$

∴ In ΔADC, DC : AD : AC = 1: $\sqrt{3}$: 2

$\Longrightarrow \frac{DC}{AC} = \frac{1}{2}$ ----------------(1)

CO is the angle bisector

∴ $\frac{CD}{AC} = \frac{DO}{OA} = \frac{1}{2}$ ----------------------(2)

$\Longrightarrow \frac{AD}{DO} = 3$ ----------------------------(3)

In ΔABC, the cevians CE, BF & AD are concurrent at O.

And EF cuts AO at G.

∴ As per Concurrency Theorem,

$\frac{AG}{GO} = \frac{AD}{DO}$ = 3 [as per (3)]

$\Longrightarrow \frac{AO}{OG} = 4$

$\Longrightarrow AO = 4OG$ ----------------- (4)

(2) & (4) ⟶

OD = 2OG

∴ $AG = GD = 3OG$

⟹ G is the midpoint of AD

$\angle ADB = \angle AKB = 90°$ (given)

∴ $ABDK$ is concyclic

The median BG of ΔABD cuts the other side of the circle at K

As per the Bisecting Chord Theorem [Page no : 47 of the Author's book "The Novelties of Geometry", available in this web site],

$2BK^2 = AB^2 + BD^2 + DK^2 + AK^2$

$\Longrightarrow BK^2 + (AB^2 - AK^2) = AB^2 + BD^2 + DK^2 + AK^2$ [$\because BK^2 = AB^2 - AK^2$]

$\Longrightarrow BK^2 = AB^2 + BD^2 + DK^2 + AK^2 - AB^2 + AK^2$

$BK^2 = BD^2 + DK^2 + 2AK^2$ ---------------------------- Proved